MIDTERM: COMMUTATIVE ALGEBRA I

Date: **13th September 2016**
The Total points is **115** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (4+4+8+8+8+8=40 points) Let R be a ring, I ideal in R and $X \subset \operatorname{Spec}(R)$. Define the subset $V(I)$ of $\operatorname{Spec}(R)$ and $\mathscr{I}(X)$. Prove or disprove the following:
  (a) $V(IJ) = V(I) \cup V(J)$ for all ideals I, J of R.
  (b) $V(\mathscr{I}(X)) = X$ for any closed subset X of $\operatorname{Spec}(R)$.
  (c) R is an integral domain then $\operatorname{Spec}(R)$ is connected in Zariski topology.
  (d) $\operatorname{Spec}(R)$ is connected in Zariski topology then R is an integral domain.

(2) (10+5=15 points) Let $0 \to A \to B \to C \to 0$ be a short excat sequence of R-modules and M be an R-module. Show that $0 \to Hom_R(C, M) \to Hom_R(B, M) \to Hom_R(A, M)$ is an exact sequence. Show that the map $Hom_R(B, M) \to Hom_R(A, M)$ need not be surjective.

(3) (5+15=20 points) Let R be a ring, S a multiplicative subset and I an ideal. Show that $S^{-1}(R/I) \cong S^{-1}R/S^{-1}I$. Let $R = k[x, y, \frac{1}{x+y}]/(xy)$. Show that $R \cong k[x, 1/x] \times k[y, 1/y]$. (Hint: Construct the natural monomorphism from $R \to k[x, 1/x] \times k[y, 1/y]$ and compute idempotents in R to show surjectivity.)

(4) (5+15=20 points) Define projective module. Let R be a local ring and M be a finitely generated projective R-module then M is a free R-module. (Do not use results whose proof you don't know.)

(5) (15 points) Let R be a ring and M be an R-module such that $M_m = 0$ for all maximal ideals m of R. Show that $M = 0$.